MULTIFRACTAL FORMALISM FOR TYPICAL PROBABILITY MEASURES ON SELF-SIMILAR SETS

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Abstract. In this work, we investigate the Hölder spectrum of typical measure (in the Baire category sense) in a general compact set and we compute the multifractal spectrum of a typical measures supported by a self-similar set. Such mesures verify the multifractal formalism.

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1. Introduction and the main result

Let K be a compact set of $\mathbb{R}^d$ endowed with the metric induced by any norm on $\mathbb{R}^d$.

The local Hölder exponent of a positive measure μ on K at $x \in K$, $h_\mu(x)$, is defined by

$$h_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

where $B(x,r)$ is the ball of center x and radius r. The purpose of the multifractal analysis of a measure μ is to investigate the singularity spectrum d_μ of μ, that is the map

$$d_\mu : h \geq 0 \mapsto \dim_{\mathcal{H}}(E_\mu(h))$$

where $E_\mu(h) = \{x \in K : \ h_\mu(x) = h\}$ and $\dim_{\mathcal{H}}$ is the Hausdorff dimension.

Generally it is very difficult to obtain the singularity spectrum directly from the definition of the Hausdorff dimension. To avoid this difficulty, the multifractal formalism provide a formula which link the singularity spectrum to the Legendre transform of mapping defined by averaged quantities of the measure, precisely to the Legendre transform of the L^q spectrum defined as follows. If j is an integer greater than 1 let $\mathcal{G}_j$ be the partition of $\mathbb{R}^d$ into dyadic boxes: $\mathcal{G}_j$ is the set of all cubes

$$I_{j,\mathbf{k}} = \prod_{i=1}^{d} \left[k_i 2^{-j}, (k_i+1)2^{-j}\right[$$

where $\mathbf{k} = (k_1, k_2, \cdots, k_d) \in \mathbb{Z}^d$.

The L^q spectrum of a measure $\mu \in \mathcal{M}(K)$ is the mapping defined for any $q \in \mathbb{R}$ by

$$\tau_\mu(q) = \liminf_{j\to+\infty} -\frac{\log \sum_{Q\in\mathcal{G}_j, \mu(Q)\neq 0} \mu(Q)^q}{j \log 2}.$$

A classical result (see for example [4]) assert that for all measure μ for all $h \geq 0$,

$$d_\mu(h) \leq (\tau_\mu)^*(h) := \inf_{q\in\mathbb{R}}(qh - \tau_\mu(q)). \tag{1}$$

A important issue in multifractal analysis is to establish when the upper bound (1) turns out to be an equality , when this happens we say that the measure μ *satisfies the multifractal formalism* at h. A lot of work has been achieved for specific measures.

In the few last years, a particular interest was allocated to generic results (in the sense of Baire or prevalence) on the space of the probability measures endowed with the weak topology or in some space of functions, see for examples [1], [2], [3], [5], [6], [11], [12].

We denote by $\mathcal{M}(K)$ the space of probability measures on K endowed with the weak topology. Recall that the weak topology on $\mathcal{M}(K)$ is induced by the metric ϱ on $\mathcal{M}(K)$ defined as follows. Let $\mathrm{Lip}(K)$ denote the family of Lipschitz functions $f : K \to \mathbb{R}$ with $|f| := \sup_{x\in K} |f(x)| \leq 1$ and $Lip(f) \leq 1$ where $Lip(f)$ denotes the Lipschitz constant of f. If μ and ν belong to $\mathcal{M}(K)$ we set

$$\varrho(\mu,\nu) = \sup\left\{\left|\int f d\mu - \int f d\nu\right| : f \in \mathrm{Lip}(K)\right\}.$$

then the space $\mathcal{M}(K)$ is complete and separable.

In [2], the authors determined the multifractal spectrum for typical measures μ (in the Baire sens) in $[0,1]^d$ and they showed that such measures satisfy the multifractal formalism. They also made the following conjecture

Conjecture 1. *For any compact set $K \subset \mathbb{R}^d$, there exists a constant $0 < D < d$ such that typical measures μ (in the Baire sens) in $\mathcal{M}(K)$ satisfy: for any $h \in [0, D]$, $d_\mu(h) = h$, and if $h > D$, $E_\mu(h) = \emptyset$.*

Whether D should be the Hausdorff dimension of K or the lower box dimension of K (or another dimension).

In this paper, we give a positive answer to this conjecture in the special case where K is a self-similar set satisfying the open set condition.

Let X be a complete metric space. We say that a set A of X is a G_δ set if it can be written as a countably intersection of dense open sets. We say that a property is typical in X if it holds on residual set, i.e. a set with complement of first Baire category. By the Baire theorem any G_δ set is dense.

Our main result is the following

Theorem 1.1. *Let K be a self-similar set satisfying the open set condition. Let s be the Hausdorff dimension of K. Then, there exists a G_δ set Ω of $\mathcal{M}(K)$ such that for all $\mu \in \Omega$,*

- *for all $h > s$, $E_\mu(h) = \emptyset$*
- *for all $h \in [0, s]$, $d_\mu(h) = h$.*

In particular, for every $q \in [0, 1]$, $\tau_\mu(q) = s(q - 1)$, and μ satisfies the multifractal formalism at every $h \in [0, s]$, i.e. $d_\mu(h) = (\tau_\mu)^*(h)$.

Our paper is organized as follows: in the second section we show, for any compact K, that for typical measure μ in $\mathcal{M}(K)$, for $h > s$, $E_\mu(h) = \emptyset$ where s is the upper box counting dimension, this can be en particular applied to self-similar sets. In the third section we recall some properties of self-similar sets that will be useful for us. Then, using the same approach as [2] with suitable modifications we prove the Theorem 1.1.

2. Results valid on compact K

Let $0 \le s < +\infty$, λ a borelian measure on K and $a \in K$. We define the lower $s-$densities of K at a with respect to λ by

$$\Theta^s_*(K, a, \lambda) = \liminf_{r \downarrow 0} (2r)^{-s} \lambda\left(K \cap B(a, r)\right).$$

In this section we will prove the following theorems.

Theorem 2.1. *Let K be a closed set of $\mathbb{R}^d$.*

(1) *Let $a \in K$. Then, there exists a G_δ set $\Omega(a)$ of $\mathcal{M}(K)$ such that for all $\mu \in \Omega(a)$, $h_\mu(a) = 0$.*
(2) *Let $s \in]0, d]$ and $A \subset K$. Assume that there exists λ a finite Borel measure on K such that for all $a \in A$*

$$\Theta^s_*(K, a, \lambda) > 0.$$

Then, there exists a G_δ set Ω of $\mathcal{M}(K)$ such that for all $\mu \in \Omega$, for all $x \in A$, $h_\mu(x) \le s$. That is, for all $\mu \in \Omega$, for all $h > s$, $E_\mu(h) \cap A = \emptyset$.

Remark 2.1. *Let $a \in K$. For all $h > 0$ the set $\Lambda_h(a) = \{\mu \in \mathcal{M}(K);\ h_\mu(a) = h\}$ is of empty interior. Indeed, if not then using the dense G_δ set $\Omega(a)$ we get $\Lambda_h(a) \cap \Omega(a) \neq \emptyset$ which is impossible.*

Let E a non-empty bounded subset or $\mathbb{R}^d$ let $N_r(E)$ be the largest number of disjoint balls of radius r with centers in E. The upper box-counting dimension of E is defined as

$$\overline{\dim}_B E = \limsup_{r \to 0} \frac{\log N_r(E)}{-\log r}.$$

Already we can prove the following result.

Theorem 2.2. *Let K be a closed set of $\mathbb{R}^d$ let $s = \overline{\dim}_B K$. Then, there exists a G_δ set Ω of $\mathcal{M}(K)$ such that for all $\mu \in \Omega$, for all $x \in K$, $h_\mu(x) \le s$. That is, for all $\mu \in \Omega$, for all $h > s$, $E_\mu(h) = \emptyset$.*

2.1. **Proof of Theorem 2.1.**

In the sequel, we will always denote by $B(x, r)$ (resp. $\overline{B}(x, r)$) the open (resp. closed) ball of center $x \in X$ and radius r, where X any metric space.

1) Let $a \in K$ and $s > 0$. Let (ν_n) be a dense sequence in $\mathcal{M}(K)$. Let $(d_n)_n$ be a decreasing sequence to 0.

Let $\theta > \dfrac{2}{s}$. Put $\beta_n = \dfrac{1}{\log|\log d_n|}$. We consider the following sequences $\alpha_n = d_n^{\beta_n}$, $r_n = d_n^{\theta s}$, $c_n = d_n^{\frac{\theta}{2} s}$. Remark that all the sequences are decreasing to 0.

Denote by

$$\mu_n = \alpha_n \delta_a + (1 - \alpha_n)\nu_n$$

where δ_a is the Dirac mass at a. Since $\rho(\mu_n, \nu_n) \leq 2\alpha_n \underset{n\to+\infty}{\to} 0$, the sequence $(\mu_n)_n$ is dense in $\mathcal{M}(K)$.

Now put

$$\Omega_N(a) = \bigcup_{k\geq N} B(\mu_k, r_k) \quad \text{and} \quad \Omega(a) = \bigcap_{N=1}^{+\infty} \Omega_N.$$

$\Omega(a)$ is a G_δ set in $\mathcal{M}(K)$ since for all N, $\Omega_N(a)$ is a dense open set.

Let $\mu \in \Omega$. There exists an increasing sequence (m_n) of integers such that for all n,

$$\rho(\mu, \mu_{m_n}) \leq r_{m_n}.$$

Since $0 < c_n \leq d_n$, we can construct a Lipschitz function f_n which satisfies $0 \leq f_n(y) \leq c_n$ for all y and $f_n(y) = c_n$ for all $y \in B(a, \frac{d_n}{2})$ and $f_n(y) = 0$ for all $y \notin B(a, d_n)$ and such that $Lip(f) \leq 1$. (For example we can consider the restriction to K of the function $f : \mathbb{R}^d \to \mathbb{R}$, $f(y) = c_n$ if $\|y - a\| \leq \frac{d_n}{2}$; $f(y) = -2\frac{c_n}{d_n}\|y-a\| + 2c_n$ if $\frac{d_n}{2} < \|y - a\| \leq d_n$; $f(y) = 0$ if $\|y - a\| > d_n$).

We have,

$$\int f_{m_n} d\mu \leq c_{m_n} \mu(B(a, d_{m_n})). \tag{2}$$

In other part, using the property of the function f we get for all n

$$\begin{aligned} \int f_{m_n} d\mu_{m_n} &\geq \alpha_{m_n} \int f_{m_n} d\delta_a \\ &\geq \alpha_{m_n} \int_{B(a, \frac{d_{m_n}}{2})} f_{m_n} d\delta_a \\ &= \alpha_{m_n} c_{m_n}. \end{aligned} \tag{3}$$

We have $\rho(\mu, \mu_{m_n}) \leq r_{m_n}$, thus using (2) and (3), for all n

$$\begin{aligned} c_{m_n} \mu(B(a, d_{m_n})) &\geq \int f_{m_n} d\mu \\ &\geq \int f_{m_n} d\mu_{m_n} - \rho(\mu, \mu_{m_n}) \\ &\geq \alpha_{m_n} c_{m_n} - r_{m_n} \end{aligned} \tag{4}$$

then by (4)

$$\begin{aligned} \mu(B(a, d_{m_n})) &\geq \alpha_{m_n} - \frac{r_{m_n}}{c_{m_n}} \\ &= d_{m_n}^{\beta_{m_n}} - d_{m_n}^{\frac{\theta}{2}s} \\ &= d_{m_n}^{\beta_{m_n}} (1 - d_{m_n}^{\frac{\theta}{2}s - \beta_{m_n}}) \end{aligned}$$

Then for n sufficiently large we get

$$\mu(B(a, d_{m_n})) \geq \frac{1}{2} d_{m_n}^{\beta_{m_n}}.$$

Finally

$$h_\mu(a) = \liminf_{r\to 0} \frac{\log\mu(B(a,r))}{\log r} \leq \liminf_{n\to+\infty} \frac{\log\mu(B(a,d_{m_n}))}{\log d_{m_n}} \leq \lim_{n\to+\infty} \beta_{m_n} = 0.$$

This finish the proof of the first point.

2) Now we must construct a dense G_δ set of $\mathcal{M}(K)$ which is independent of $a \in A$.

Let (ν_n) be a dense sequence in $\mathcal{M}(K)$. Let $\nu = \dfrac{1}{\gamma(K)}\gamma\llcorner K$, (remark that since $\Theta^s_*(K,a,\lambda) > 0$ for some point a then $\lambda(K) > 0$).

Let $(\alpha_n)_n$ be a sequence decreasing to 0. Denote by

$$\mu_n = \alpha_n\nu + (1-\alpha_n)\nu_n.$$

Since $\rho(\mu_n,\nu_n) \leq 2\alpha_n \underset{n\to+\infty}{\to} 0$, the sequence $(\mu_n)_n$ is dense in $\mathcal{M}(K)$.

Let $\theta > 1 + \dfrac{2}{s}$. We consider the following sequences $d_n = \exp\left(-\dfrac{1}{\alpha_n}\right)$, $r_n = d_n^{\theta s}$ and $c_n = d_n^{\frac{\theta-1}{2}s}$. All the defined sequences are decreasing to 0. Remark that $\alpha_n = d_n^{\beta_n}$ with $\lim_{n\to\infty}\beta_n = 0$.

Now we set

$$\Omega_N = \bigcup_{k\geq N} B(\mu_k, r_k) \quad \text{and} \quad \Omega = \bigcap_{N=1}^{+\infty}\Omega_N.$$

Ω is a G_δ set in $\mathcal{M}(K)$ since for all N, Ω_N is a dense open set.

Let $\mu \in \Omega$. There exists an increasing sequence (m_n) of integers such that for all n,

$$\rho(\mu,\mu_{m_n}) \leq r_{m_n}.$$

Let $a \in A$. By our hypothesis, there exist $c > 0$ and $v > 0$, such that

$$\text{if } 0 < r < v, \quad \nu(B(a,r)) \geq cr^s. \tag{5}$$

Let f_n be the Lipschitz function as constructed in the first point of the theorem associated to the newer sequences (c_n) and (d_n).

We get for all n,

$$\int f_{m_n} d\mu \leq c_{m_n}\mu(B(a,d_{m_n})). \tag{6}$$

In other part, using the property of the function f we get for all n such that $d_n \leq v$,

$$\begin{aligned}
\int f_{m_n} d\mu_{m_n} &\geq \alpha_{m_n}\int f_{m_n} d\nu \\
&\geq \alpha_{m_n}\int_{B(a,\frac{d_{m_n}}{2})} f_{m_n} d\nu \\
&= \alpha_{m_n} c_{m_n}\nu\left(B(a,\frac{d_{m_n}}{2})\right) \\
&\geq c'\alpha_{m_n} c_{m_n} d^s_{m_n}. \qquad (7)
\end{aligned}$$

Then by (6) and (7) we get

$$\begin{aligned}\mu(B(a,d_{m_n})) &\geq c'\alpha_{m_n}d_{m_n}^s-\frac{r_{m_n}}{c_{m_n}}\\ &= c'd_{m_n}^{\beta_{m_n}+s}-d_{m_n}^{\frac{\theta+1}{2}s}\\ &= d_{m_n}^{\beta_{m_n}+s}(c'-d_{m_n}^{\frac{\theta-1}{2}s-\beta_{m_n}})\end{aligned}$$

Then for n sufficiently large we get

$$\mu(B(a,d_{m_n}))\geq\frac{c'}{2}d_{m_n}^{\beta_{m_n}+s}.$$

Finally

$$h_\mu(a)=\liminf_{r\to 0}\frac{\log\mu(B(a,r))}{\log r}\leq\liminf_{n\to+\infty}\frac{\log\mu(B(a,d_{m_n}))}{\log d_{m_n}}\leq\lim_{n\to+\infty}s+\beta_{m_n}=s.$$

This finish the proof of the second point.

2.2. **Proof of Theorem 2.2.** For all $n\in\mathbb{N}$, we put $L_n=N_{2^{-n}}(K)$ and let $a_{1,n},\dots,a_{L_n,n}$ be L_n points of K such that for $i\neq j$

$$B(a_{i,n},2^{-n})\cap B(a_{j,n},2^{-n})=\emptyset.$$

Let $\alpha_n=2^{-\sqrt{n}}$. Let (ν_n) be a dense sequence in $\mathcal{M}(K)$. We consider the probability measures

$$\Pi_n=L_n^{-1}\sum_{i=1}^{L_n}\delta_{a_{i,n}}$$

and

$$\mu_n=\alpha_n\Pi_n+(1-\alpha_n)\nu_n.$$

Since $\rho(\mu_n,\nu_n)\leq 2\alpha_n\underset{n\to+\infty}{\to}0$, the sequence $(\mu_n)_n$ is dense in $\mathcal{M}(K)$.

Now put $r_n=2^{-(s+2)n}$. We set

$$\Omega_N=\bigcup_{k\geq N}B(\mu_k,r_k)\quad\text{and}\quad\Omega=\bigcap_{N=1}^{+\infty}\Omega_N.$$

Ω is a G_δ set in $\mathcal{M}(K)$ since for all N, Ω_N is a dense open set.

Let $\mu\in\Omega$. There exists an increasing sequence (m_n) of integers such that for all n,

$$\rho(\mu,\mu_{m_n})\leq r_{m_n}.$$

Let $x\in K$. Since L_{m_n} is the largest number of disjoint balls of radius 2^{-m_n} with centers in K then there exists $i\in\{1,\dots,L_{m_n}\}$ such that

$$B(x,2^{-m_n})\cap B(a_{i,m_n},2^{-m_n})\neq\emptyset.$$

Thus $a_{i,m_n}\in B(x,22^{-m_n})$.

Let $f_n\in Lip(K)$ such that for all $y\in B(x,22^{-m_n})$, $f_n(y)=2^{-m_n}$, for all $y\notin B(x,42^{-m_n})$, $f_n(y)=0$ and $0\leq f_n\leq 2^{-m_n}$.

We get for all n,

$$\int f_n d\mu\leq 2^{-m_n}\mu(B(x,42^{-m_n})).\tag{8}$$

In other part, using the property of the function f_n we get for all n,

$$\begin{aligned}\int f_n d\mu_{m_n} &\geq \alpha_{m_n}\int f_n d\Pi_{m_n}\\ &\geq \alpha_{m_n}\int_{B(x,22^{-m_n})} f_n d\Pi_{m_n}\\ &\geq \alpha_{m_n}L_{m_n}^{-1}\int_{B(x,22^{-m_n})} f_n d\delta_{a_{i,m_n}}\\ &= \alpha_{m_n}2^{-m_n}L_{m_n}^{-1}.\end{aligned}$$

Let t such that $\overline{\dim}_B K = s < t < s+1$. Then there exists $v > 0$ such that for all $0 < r < v$,

$$N_r(K) < r^{-t}.$$

Thus, for n sufficiently large such that $2^{-m_n} < v$ we get $L_{m_n}^{-1} > 2^{-tm_n}$. Then

$$\int f_n d\mu_{m_n} \geq \alpha_{m_n}2^{-m_n}2^{-tm_n}. \tag{9}$$

We have $\rho(\mu,\mu_{m_n}) \leq r_{m_n}$, thus using (8) and (9), we get for n sufficiently large

$$\begin{aligned}2^{-m_n}\mu(B(x,42^{-m_n})) &\geq \int f_n d\mu\\ &\geq \int f_n d\mu_{m_n} - \rho(\mu,\mu_{m_n})\\ &\geq \alpha_{m_n}2^{-m_n}2^{-tm_n} - r_{m_n} \qquad (10)\end{aligned}$$

then by (10)

$$\begin{aligned}\mu(B(x,42^{-m_n})) &\geq \alpha_{m_n}2^{-tm_n} - 2^{m_n}r_{m_n}\\ &= 2^{-tm_n(1+\frac{1}{t\sqrt{m_n}})} - 2^{-(s+1)m_n}\\ &= 2^{-tm_n(1+\frac{1}{t\sqrt{m_n}})}(1-2^{(t-(s+1))m_n+\sqrt{m_n}})\end{aligned}$$

$\lim_{n\to+\infty}(t-(s+1))m_n + \sqrt{m_n} = -\infty$, then for n sufficiently large we get

$$\mu(B(x,42^{-m_n})) \geq \frac{1}{2}2^{-tm_n(1+\frac{1}{t\sqrt{m_n}})}.$$

Finally

$$\begin{aligned}h_\mu(x) &= \liminf_{r\to 0}\frac{\log\mu(B(x,r))}{\log r} \leq \liminf_{n\to+\infty}\frac{\log\mu(B(x,42^{-m_n}))}{\log 42^{-m_n}}\\ &= \liminf_{n\to+\infty}\frac{\log\mu(B(x,42^{-m_n}))}{\log 2^{-m_n}}\\ &\leq \lim_{n\to+\infty} t + \frac{1}{\sqrt{m_n}} = t.\end{aligned}$$

Then, for all t such that $s = \overline{\dim}_B K < t < s+1$, we have $h_\mu(x) \leq t$. Thus $h_\mu(x) \leq s$.

We conclude that for all $\mu \in \Omega$, for all $x \in K$, $h_\mu(x) \leq s$.

3. The mulitifractal spectrum of typical measures on self-similar sets

In this section we focus on the special case where the compact K is a self-similar set. We recall the definition of such set and some related metric facts that will be useful for our purpose.

3.1. **Recalls on self-similar sets.** We refer the reader to [7], [8], [10] for more properties of self-similar sets.

By $\mathbb{R}^d$, $d \geq 1$, we denote the $d-$dimensional Euclidean space, by $\mathcal{B}(x,r)$ the balls $\{y : |x-y| < r\}$, $x \in \mathbb{R}^d$, $r > 0$, $|\,|$ the canonical Euclidean norm. Let $\mathbf{S} = \{S_1, \cdots, S_p\}$ be a given set of contractive similitudes, that is

$$|S_i(x) - S_i(y)| = \alpha_i \, |x-y| \tag{11}$$

$i = 1, \ldots, p$, where we assume that $0 < \alpha_1 \leq \cdots \leq \alpha_p < 1$. We will call briefly S_i an α_i-similitude.

We use the classical following notations: For $n \in \mathbb{N}^*$ we note $\mathbb{A}_n = \{\mathbf{i} = i_1 \cdots i_n : \forall\, k \in \{1, ..., n\},\ i_k \in \{1, ..., p\}\}$ the sets of words of length n in the alphabet $\{1, ..., p\}$. $\mathbb{A}^* = \bigcup\limits_{n \geq 1} \mathbb{A}_n$. Finally $\mathbb{A} = \{\mathbf{i} = i_1 \cdots i_k \cdots :\ i_k \in \{1, ..., p\}\}$ the set of infinite words.

Without confusion we note in bold characters the elements of $\mathbb{A}$ and $\mathbb{A}^*$. For $\mathbf{i} = i_1 \cdots i_n \in \mathbb{A}_n$, we set $|\mathbf{i}| = n$ the length of $\mathbf{i}$. For $\mathbf{i} \in \mathbb{A}$ and $n \geq 1$, we note $\mathbf{i}[n] = i_1 \cdots i_n$.

If $\mathbf{i} = i_1 \cdots i_n \in \mathbb{A}_n$, then $S_{\mathbf{i}} := S_{i_1} \circ \cdots \circ S_{i_n}$ is a contraction with ratio $\alpha_{\mathbf{i}} = \alpha_{i_1} \cdots \alpha_{i_n}$. If T is any subset of $\mathbb{R}^d$, then $T_{\mathbf{i}} = S_{\mathbf{i}}(T)$ with the convention $T_\emptyset = T$ and $S_\emptyset =$Id. Particularly, for $\mathbf{i} \in \mathbb{A}_n$ $K_{\mathbf{i}}$ is called an $n-$complex.

We say that the self similar K satisfy the open set condition, if there exists an open set U such that for all $i \in \{1, \cdots, p\}$,

$$S_i(U) \subset U \quad \text{and} \quad S_i(U) \cap S_j(U) = \emptyset \ \text{ if } i \neq j.$$

Let $R > 0$. We set

$$I(R) = \left\{\mathbf{i} = i_1 \cdots i_n \in \mathbb{A}^* \ : \ \alpha_{\mathbf{i}} \leq R < \alpha_{\mathbf{i}[n-1]}\right\}.$$

Let s the real defined by

$$\sum_{k=1}^{p} \alpha_k^s = 1.$$

Under the open set condition we have $\dim_{\mathcal{H}}(K) = s$ and $0 < \mathcal{H}^s(K) < +\infty$ where $\mathcal{H}^s$ is the $s-$Hausdorff measure (see for example [4]). We set

$$\lambda = \frac{1}{\mathcal{H}(K)} \mathcal{H}^s \llcorner K.$$

In the sequel we denote by $\sharp A$ the cardinality of the set A.

We gather the useful properties for us in the following proposition (see [10], [13]. Some results are also in the proof of the Theorem 9.3 in [4]).

Proposition 3.1. *We assume that the open set condition is satisfied with the open set U. Let $s = dim_{\mathcal{H}}(K)$.*

(1) *There exist two constants* $c_1, c_2 > 0$*, such that for all* $R > 0$*,*

$$c_1 R^{-s} \leq \sharp I(R) \leq c_2 R^{-s}.$$

(2) *For all* $R > 0$*,*

$$\sum_{\mathbf{i} \in I(R)} \alpha_{\mathbf{i}} = 1.$$

(3) *There exist two constants* $c_1, c_2 > 0$ *such that for all* $R > 0$*, for all* $x \in K$

$$c_1 R^s \leq \lambda\left(B(x, R)\right) \leq c_2 R^s.$$

(4) *Let* $R > 0$*.*
   (a) $K = \bigcup_{\mathbf{i} \in I(R)} K_{\mathbf{i}}$*.*
   (b) *For all* $\mathbf{i}, \mathbf{j} \in I(R)$ *such that* $\mathbf{i} \neq \mathbf{j}$*, we have*

$$U_{\mathbf{i}} \cap U_{\mathbf{j}} = \emptyset \quad \textit{and} \quad \lambda\left(K_{\mathbf{i}} \cap K_{\mathbf{j}}\right) = 0.$$

3.2. **Proof of Theorem 1.1.** Let K be a self-similar set associated to the system $\mathbf{S} = \{S_1, \cdots, S_p\}$ of α_i−simiitudes satisfying the open set condition, where we assume that $0 < \alpha_1 \leq \cdots \leq \alpha_p < 1$. We adopt the notations of the previous section. Denote by $s = \dim_{\mathcal{H}} K$.

Since $s = \overline{\dim}_B K$, then by Theorem 2.2, we know that there exists a G_δ set Ω' of $\mathcal{M}(K)$ such that for all $\mu \in \Lambda$, for all $x \in K$, $h_\mu(x) \leq s$.

To achieve the proof of the Theorem 1.1, we will prove that there exists a G_δ set Ω'' of $\mathcal{M}(K)$ such that for all $\mu \in \Omega''$, for all $h \in]0, s]$, $d_\mu(h) = h$. Then to recover $h = 0$, we fix any point $x_0 \in K$ and we consider the G_δ set $\Omega(x_0)$ associated to x_0 in Theorem 2.1. We consider finally $\Omega = \Omega' \cap \Omega'' \cap \Omega(x_0)$ which stills a G_δ set of $\mathcal{M}(K)$.

Remark 3.1. *In our proofs many constants will appear with no importance. To relieve the work, we will sometimes denote the constants by the same letter between consecutive inequalities even if the constants are different.*

We adopt the same approach of [2] with suitable modifications.

For any $\mathbf{i} \in I(2^{-J_N})$ we pick an $x_{\mathbf{i}} \in K_{\mathbf{i}}$. The family of point $\bigcup_N \left\{x_{\mathbf{i}} : \ \mathbf{i} \in I(2^{-J_N})\right\}$ will be fixed in the rest of the paper.

We define the following probability measure

$$\lambda_n = \sum_{\mathbf{i} \in I(2^{-J_n})} \alpha_{\mathbf{i}}^s \delta_{x_{\mathbf{i}}}$$

where $\delta_{x_{\mathbf{i}}}$ is the Dirac mass at the point $x_{\mathbf{i}}$. λ_n is probability measure since $\sum_{\mathbf{i} \in I(2^{-J_n})} \alpha_{\mathbf{i}}^s = 1$ (see Proposition 3.1), and is supported by K.

Let $\beta_n = \dfrac{J_n}{n}$ and we denote by

$$\mu_n = \beta_n \lambda_n + (1 - \beta_n) \nu_n$$

the sequence $(\mu_n)_n$ is dense sequence in $\mathcal{M}(K)$ since $\varrho(\mu_n, \nu_n) \leq 2\beta_n$.

Definition 3.1. *Let $n \in \mathbb{N}^*$. We introduce*
$$\Omega_n = \bigcup_{k \geq n} B(\mu_k, 2^{-sJ_k^2}) \quad \textit{and} \quad \Omega'' = \bigcap_{n \geq 1} \Omega_n$$

Ω'' is a G_δ set of $\mathcal{M}(K)$.

Let $\mu \in \Omega''$ be fixed. There exists a sequence $(J_{N_p})_{p \geq 1}$ such that for all p,
$$\varrho(\mu, \mu_{N_p}) < 2^{-sJ_{N_p}^2}.$$

Definition 3.2. *Let $\theta \geq 1$. Let us introduce the set of points*
$$\Lambda_{\theta,p} = \bigcup_{\mathbf{i} \in I(2^{-J_{N_p}})} \overline{B}(x_{\mathbf{i}}, 2^{-\theta J_{N_p}}) \cap K.$$
and then let us define
$$\Lambda_\theta = \bigcap_{P \geq 1} \bigcup_{p \geq P} \Lambda_{\theta,p}.$$

Lemma 3.1. *Let $\epsilon > 0$. There exist p_ϵ and $c > 0$, such that for all $p \geq p_\epsilon$ and for all $x \in \Lambda_{\theta,p}$,*
$$\mu(B(x, 22^{-\theta J_{N_p}})) \geq c 2^{-s(1+\epsilon)J_{N_p}}. \tag{12}$$

Proof. Let $\epsilon > 0$ and $x \in \Lambda_{\theta,p}$. Then there exists $\mathbf{i} \in I(2^{-J_{N_p}})$ such that $x_{\mathbf{i}} \in \overline{B}(x, 2^{-\theta J_{N_p}})$. Thus
$$\mu_{N_p}(\overline{B}(x, 2^{-\theta J_{N_p}})) \geq \beta_{N_p} \alpha_{\mathbf{i}}^s \geq c_1 \beta_{N_p} 2^{-sJ_{N_p}} = c_1 2^{-s(1+\frac{1}{sN_p})J_{N_p}} \geq c_1 2^{-s(1+\epsilon)J_{N_p}}$$
for p so large that $\dfrac{1}{sN_p} < \epsilon$.

Let $f_{\theta,p}$ be a lipschitz function on K with $f \in \mathrm{Lip}(K)$ such that for all $z \in \overline{B}(x, 2^{-\theta J_{N_p}})$, $f_{\theta,p}(z) = 2^{-\theta J_{N_p}}$, for all $z \notin \overline{B}(x, 22^{-\theta J_{N_p}})$, $f_{\theta,p}(z) = 0$, and $0 \leq f_{\theta,p} \leq 2^{-\theta J_{N_p}}$. By construction,
$$\int f_{\theta,p} d\mu \leq 2^{-\theta J_{N_p}} \mu(B(x, 22^{-\theta J_{N_p}}))$$
and
$$\int f_{\theta,p} d\mu_{N_p} \geq c_1 2^{-\theta J_{N_p}} 2^{-s(1+\epsilon)J_{N_p}}$$
thus
$$\begin{aligned} 2^{-\theta J_{N_p}} \mu(B(x, 22^{-\theta J_{N_p}})) &\geq \int f_{\theta,p} d\mu \\ &\geq \int f_{\theta,p} d\mu_{N_p} - \varrho(\mu, \mu_{N_p}) \\ &\geq c_1 2^{-\theta J_{N_p}} 2^{-s(1+\epsilon)J_{N_p}} - 2^{-sJ_{N_p}^2} \end{aligned}$$
when p is sufficiently large
$$2^{-sJ_{N_p}^2} \leq \frac{1}{2} c_1 2^{-\theta J_{N_p}} 2^{-s(1+\epsilon)J_{N_p}}$$
thus there exists p_ϵ such that for all $p \geq p_\epsilon$,
$$\mu(B(x, 22^{-\theta J_{N_p}})) \geq \frac{1}{2} c_1 2^{-s(1+\epsilon)J_{N_p}}.$$

□

Proposition 3.2. *Let $\theta \geq 1$ and $x \in \Lambda_\theta$. Then $h_\mu(x) \leq \dfrac{s}{\theta}$.*

Proof. If $x \in \Lambda_\theta$, then (12) is satisfied for infinite number of integer p. Hence, for all $\epsilon > 0$, there is a sequence of infinite real numbers (r_p) decreasing to 0 such that for all p

$$\mu(B(x, 2r_p)) \geq c r_p^{\frac{s}{\theta}(1+\epsilon)}$$

this implies that $h_\mu(x) \leq \dfrac{s}{\theta}(1+\epsilon)$ for all $\epsilon > 0$, the result follows. □

Proposition 3.3. *For all $\theta \geq 1$, $dim_{\mathcal{H}} \Lambda_\theta \leq \dfrac{s}{\theta}$.*

Proof. The result is obvious when $\theta = 1$, since $\Lambda \subset K$ and $\dim_H(K) = s$.

Let $\theta > 1$ and $t > \dfrac{s}{\theta}$. For all $P \geq 1$, Λ_θ is covered by $\bigcup_{p \geq P} \Lambda_{\theta,p}$. Hence, for any $\delta > 0$ and using the fact that $\sharp I(R) \leq cR^{-s}$ (see [13]) we obtain

$$\begin{aligned}
\mathcal{H}^t_\delta(\Lambda_\theta) &\leq \mathcal{H}^t_\delta\left(\bigcup_{p \geq P} \Lambda_{\theta,p}\right) \\
&\leq \sum_{p \geq P} \sum_{\mathbf{i} \in \mathbf{I}(\mathbf{2^{-J_{N_p}}})} \left|\overline{B}(x_{\mathbf{i}}, 2^{-\theta J_{N_p}})\right|^t \\
&\leq c \sum_{p \geq P} 2^{-t\theta J_{N_p}} \sharp I(2^{-J_{N_p}}) \\
&\leq c \sum_{p \geq P} 2^{-t\theta J_{N_p}} 2^{s J_{N_p}}
\end{aligned}$$

since $t > \dfrac{s}{\theta}$, this series is convergent. Hence, $\mathcal{H}^t_\delta(\Lambda_\theta) \leq c \lim_{P \to \infty} \sum_{p \geq P} 2^{-t\theta J_{N_p}} 2^{s J_{N_p}} = 0$. Thus, $\mathcal{H}^t(\Lambda_\theta) = \lim_{\delta \to 0} \mathcal{H}^t_\delta(\Lambda_\theta) = 0$. This implies that $\dim_H(\Lambda_\theta) \leq t$ for all $t > \dfrac{s}{\theta}$, then we conclude. □

Let m be any Borel measure on K. The Hausdorff dimension of m is defined by

$$\dim_{\mathcal{H}} m = \inf\left\{\dim_{\mathcal{H}} E \ : \ E \subset K, \ m(E) > 0\right\}.$$

When $m(E) > 0$ then $\dim_{\mathcal{H}} E \geq \dim_{\mathcal{H}} m$. In other words,

$$\text{if } \dim_{\mathcal{H}} E < \dim_{\mathcal{H}} m, \text{ then } m(E) = 0. \tag{13}$$

As in [2] we have the following result

Theorem 3.1. *For every $\theta \geq 1$, there is a measure m_θ supported in Λ_θ, a constant $C > 0$ and a positive sequence (η_p) decreasing to 0 such that for every Borel set B,*

$$\textit{if } |B| \leq \eta_p, \quad m_\theta(B) \leq C\,|B|^{\frac{s}{\theta} - \frac{2}{p-1}}. \tag{14}$$

In particular, $dim_{\mathcal{H}} m_\theta \geq \dfrac{s}{\theta}$.

Proof. We will construct a suitable Cantor set $\mathcal{C}_\theta$ included in Λ_θ and a measure m_θ supported on $\mathcal{C}_\theta$ with monofractal behaviour.

We suppose that the sequence (J_{N_p}) is sufficiently rapidly decreasing. Precisely, assume that

$$J_{N_{p+1}} > \max((p+1)\theta J_{N_p}, e^{J_{N_p}}). \tag{15}$$

The following lemma will be useful for us to control the cardinality of some sets of balls.

Lemma 3.2. *Let $R > 0$, $\mathbf{i} \in I(R)$. For every $c > 0$, there exists $M \in \mathbb{N}^*$ independent of R such that*

$$\sharp \left\{ \mathbf{j} \in I(R);\ \overline{B}(x_{\mathbf{i}}, cR) \cap \overline{B}(x_{\mathbf{j}}, cR) \neq \emptyset \right\} \leq M. \tag{16}$$

Proof. In the sequel we denote by λ the measure associated to the self similar K, see Proposition 3.1 for its properties.

Denote by

$$T_{\mathbf{i}} = \sharp \left\{ \mathbf{j} \in I(R);\ \overline{B}(x_{\mathbf{i}}, cR) \cap \overline{B}(x_{\mathbf{j}}, cR) \neq \emptyset \right\}.$$

Since $\overline{B}(x_{\mathbf{i}}, cR) \cap \overline{B}(x_{\mathbf{j}}, cR) \neq \emptyset$ and $|K_{\mathbf{j}}| = |K|\, \alpha_{\mathbf{i}} \leq |K|\, R$ then there exists a constant a independent of R such that

$$K_{\mathbf{j}} \subset B(x_{\mathbf{i}}, aR).$$

Hence

$$T_{\mathbf{i}} \leq \sharp \left\{ \mathbf{j} \in I(R) :\ K_{\mathbf{j}} \subset B(x_{\mathbf{i}}, aR) \cap K \right\}.$$

It follows that

$$\lambda \left(\bigcup_{\mathbf{j};\ K_{\mathbf{j}} \subset B(x_{\mathbf{i}}, aR)} K_{\mathbf{j}} \right) \leq \lambda \left(B(x_{\mathbf{i}}, aR) \cap K \right) \leq cR^s.$$

Since for $\mathbf{j} \neq \mathbf{j}' \in I(R)$ $\lambda\left(K_{\mathbf{j}} \cap K_{\mathbf{j}'}\right) = 0$, we obtain

$$\sum_{\mathbf{j} \in I(R):\ K_{\mathbf{j}} \subset B(x_{\mathbf{i}}, aR)} \lambda(K_{\mathbf{j}}) \leq cR^s.$$

We know that for all $\mathbf{j} \in I(R)$, $c' R^s \leq \lambda(K_{\mathbf{j}})$, which gives

$$c' \sharp \left\{ \mathbf{j} \in I(R) :\ K_{\mathbf{j}} \subset B(x_{\mathbf{i}}, aR) \cap K \right\} R^s \leq cR^s.$$

Then

$$\sharp \left\{ \mathbf{j} \in I(R) :\ K_{\mathbf{j}} \subset B(x_{\mathbf{i}}, aR) \cap K \right\} \leq \frac{c}{c'}.$$

Since $T_{\mathbf{i}} \leq \sharp \left\{ \mathbf{j} \in I(R) :\ K_{\mathbf{j}} \subset B(x_{\mathbf{i}}, aR) \cap K \right\}$ we get the desired result. □

Denote by $\widetilde{F}_1$ a set formed by the largest number of disjoint balls $B(x_{\mathbf{i}}, 2^{-J_{N_1}})$, $\mathbf{i} \in I(2^{-J_{N_1}})$. We denote by

$$D_1 = \left\{ \mathbf{i} \in I(2^{-J_{N_1}}) :\ x_{\mathbf{i}} \text{ is a center of ball in } \widetilde{F}_1 \right\}.$$

Then, we set

$$F_1 = \left\{ \overline{B}(x_{\mathbf{i}}, 2^{-\theta J_{N_1}}) \cap K :\ \mathbf{i} \in D_1 \right\}.$$

We denote by $\Delta_1 = \sharp F_1$. Remark that $\sharp F_1 = \sharp \widetilde{F}_1$. Using the Lemma 3.2, we get

$$\frac{\sharp I(2^{-J_{N_1}})}{M} \leq \Delta_1 \leq \sharp I(2^{-J_{N_1}}).$$

Hence

$$c\frac{2^{sJ_{N_1}}}{M} \le \Delta_1 \le c' 2^{sJ_{N_1}}.$$

We define a probability measure m_1 by giving the value $m_1(V) = \dfrac{1}{\Delta_1}$ for each element $V \in F_1$ and then we extend m_1 to a Borel probability measure on the algebra generated by F_1, i.e. on $\sigma(V : V \in F_1)$.

Assume that we have constructed $F_1, \ldots, F_p$, $p \ge 1$ and a measure m_p on the algebra $\sigma(V : V \in F_p)$. Let $V \in F_p$. There exists $\mathbf{i} \in I(2^{-J_{N_p}})$ such that $V = \overline{B}(x_{\mathbf{i}}, 2^{-\theta J_{N_p}}) \cap K$.

Let $r_p = 2^{-\theta J_{N_p}} - |K|\, 2^{-J_{N_{p+1}}}$. We have $\dfrac{1}{2} 2^{-\theta J_{N_p}} \le r_p \le 2^{-\theta J_{N_p}}$. Let us consider

$$D_{p,\mathbf{i}} = \left\{ \mathbf{j} \in I(2^{-J_{N_{p+1}}}) : \ K_{\mathbf{j}} \cap B(x_{\mathbf{i}}, r_p) \neq \emptyset \right\}.$$

Lemma 3.3. *There exist two constants $c_1, c_2 > 0$ such that for all p*

$$c_1 2^{-s\theta J_{N_p}} 2^{sJ_{N_{p+1}}} \le \sharp D_{p,\mathbf{i}} \le c_2 2^{-s\theta J_{N_p}} 2^{sJ_{N_{p+1}}}. \tag{17}$$

Proof. Recall that $K = \displaystyle\bigcup_{\mathbf{j} \in I(2^{-J_{N_{p+1}}})} K_{\mathbf{j}}$. Then,

$$\begin{aligned} B(x_{\mathbf{i}}, r_p) \cap K &= \bigcup_{\mathbf{j} \in I(2^{-J_{N_{p+1}}})} B(x_{\mathbf{i}}, r_p) \cap K_{\mathbf{j}} \\ &= \bigcup_{\mathbf{j} \in D_{p,\mathbf{i}}} B(x_{\mathbf{i}}, r_p) \cap K_{\mathbf{j}} \end{aligned}$$

thus

$$\begin{aligned} \lambda\left(B(x_{\mathbf{i}}, r_p) \cap K\right) &= \lambda\left(\bigcup_{\mathbf{j} \in D_{p,\mathbf{i}}} B(x_{\mathbf{i}}, r_p) \cap K_{\mathbf{j}} \right) \\ &\le \sum_{\mathbf{j} \in D_{p,\mathbf{i}}} \lambda(K_{\mathbf{j}}) \\ &\le c \sharp D_{p,\mathbf{i}} 2^{-sJ_{N_{p+1}}} \end{aligned}$$

But $\lambda\left(B(x_{\mathbf{i}}, r_p) \cap K\right) \ge c' r_p^s \ge c'' 2^{-s\theta J_{N_p}}$, hence there exists c_1 such that

$$c_1 2^{-s\theta J_{N_p}} 2^{sJ_{N_{p+1}}} \le \sharp D_{p,\mathbf{i}}.$$

In the other hand, for $\mathbf{j} \in D_{p,\mathbf{i}}$, $K_{\mathbf{j}} \cap B(x_{\mathbf{i}}, r_p) \neq \emptyset$, this implies that $K_{\mathbf{j}} \subset B(x_{\mathbf{i}}, 2^{-\theta J_{N_p}})$ (since $|K_{\mathbf{j}}| \le |K|\, 2^{-J_{N_{p+1}}}$). Thus,

$$\bigcup_{\mathbf{j} \in D_{p,\mathbf{i}}} K_{\mathbf{j}} \subset B(x_{\mathbf{i}}, 2^{-\theta J_{N_p}}) \cap K$$

hence

$$\begin{aligned} \lambda\left(\bigcup_{\mathbf{j} \in D_{p,\mathbf{i}}} K_{\mathbf{j}} \right) &\le \lambda\left(B(x_{\mathbf{i}}, 2^{-\theta J_{N_p}}) \cap K\right) \\ &\le c 2^{-s\theta J_{N_p}} \end{aligned}$$

But, $\lambda\left(\bigcup_{\mathbf{j}\in D_{p,\mathbf{i}}} K_{\mathbf{j}}\right) = \sum_{\mathbf{j}\in D_{p,\mathbf{i}}} \lambda\left(K_{\mathbf{j}}\right)$ (since $\lambda\left(K_{\mathbf{j}}\cap K_{\mathbf{j}'}\right)=0$ for $\mathbf{j}\neq\mathbf{j}'\in I(2^{-J_{N_{p+1}}})$).
Thus

$$\sum_{\mathbf{j}\in D_{p,\mathbf{i}}} \lambda\left(K_{\mathbf{j}}\right) \leq c2^{-s\theta J_{N_p}}$$

but

$$\sum_{\mathbf{j}\in D_{p,\mathbf{i}}} \lambda\left(K_{\mathbf{j}}\right) \geq c'\sharp D_{p,\mathbf{i}} 2^{-sJ_{N_{p+1}}}$$

thus, there exists c_2 such that

$$\sharp D_{p,\mathbf{i}} \leq c_2 2^{-s\theta J_{N_p}} 2^{sJ_{N_{p+1}}}.$$

$\square$

Let us define the set $\widetilde{F}_{p+1}(V)$ formed by the largest number of balls $B(x_{\mathbf{j}}, 2^{-\theta J_{N_{p+1}}})$: $\mathbf{j}\in D_{p,\mathbf{i}}$ such that if $\mathbf{j}\neq\mathbf{j}'\in D_{p,\mathbf{i}}$ we have

$$B(x_{\mathbf{j}}, 2^{-J_{N_{p+1}}}) \cap B(x_{\mathbf{j}'}, 2^{-J_{N_{p+1}}}) = \emptyset.$$

Then, we set

$$F_{p+1}(V) = \left\{\overline{U}\cap K : \ U\in\widetilde{F}_{p+1}(V)\right\}.$$

Remark that for all $U\in F_{p+1}(V)$, $U\subset V=\overline{B}(x_{\mathbf{i}}, 2^{-\theta J_{N_p}})\cap K$.

We have the following lemma

Lemma 3.4. *There exist two constants $c_1', c_2' > 0$ such that for all p*

$$c_1' 2^{-s\theta J_{N_p}} 2^{sJ_{N_{p+1}}} \leq \sharp F_{p+1}(V) \leq c_2' 2^{-s\theta J_{N_p}} 2^{sJ_{N_{p+1}}}. \tag{18}$$

Proof. We have $\sharp F_{p+1}(V) = \sharp\widetilde{F}_{p+1}(V)$. Since $\sharp\widetilde{F}_{p+1}(V)\leq\sharp D_{p,\mathbf{i}}$, we get obviously the second inequality.

Using the lemma 3.2 we can pick at least $\frac{c'}{M}\sharp D_{p,\mathbf{i}}$ element of $\widetilde{F}_{p+1}(V)$ such that the balls of radius $2^{-J_{N_p}}$ are disjoint. Since $\widetilde{F}_{p+1}(V)$ is of largest cardinality then we conclude that

$$\frac{c'}{M}\sharp D_{p,\mathbf{i}} \leq \widetilde{F}_p(V)$$

and then we get the first inequality by using the lemme 3.3. $\square$

Now we define $F_{p+1} = \bigcup_{V\in F_p} F_p(V)$. We define a probability measure m_{p+1} by giving the mass $m_{p+1}(U) = \dfrac{m_p(V)}{\sharp F_{p+1}(V)}$, where V is the unique element of F_p containing U. We extend then m_{p+1} to $\sigma(U: \ U\in F_{p+1})$.

Finally we set

$$\mathcal{C}_\theta = \bigcap_{p\geq 1}\bigcup_{V\in F_p} V.$$

By the Kolmogorov extension theorem, $(m_p)_{p\geq 1}$ converges weakly to a Borel probability measure m_θ supported on $\mathcal{C}_\theta$ and such that for every $p\geq 1$, for every $V\in F_p$, $m_\theta(V) = m_p(V)$.

3.2.1. *Hausdorff dimension of $\mathcal{C}_\theta$ and m_θ.* As is [2] we first prove that m_θ has an almost monofractal behavior on set belonging to $\bigcup_p F_p$.

Lemma 3.5. *When p is sufficiently large, for every $V \in F_p$*

$$2^{-sJ_{N_p}(1+\frac{1}{p})} \leq m_\theta(V) \leq 2^{-sJ_{N_p}(1-\frac{2}{p})} \tag{19}$$

and

$$|V|^{\frac{s}{\theta}+\frac{1}{|\log|V||}} \leq m_\theta(V) \leq |V|^{\frac{s}{\theta}-\frac{1}{|\log|V||}}. \tag{20}$$

Proof. Let $V \in F_p$. We denote $\Delta_{p+1}(V) = \sharp F_{p+1}(V)$ (recall that $F_{p+1}(V)$ is the set of element of F_{p+1} included in V). For $k \leq p$ denote by V_k the unique element in F_k containing V. By construction of the measure m_θ we obtain

$$m_\theta(V) = \left(\prod_{k=1}^{p} \Delta_k(V_{k-1})\right)^{-1}.$$

Using Lemma 3.4, there exist two constants $c_1', c_2' > 0$ such that

$$c_1' 2^{-s\theta J_{N_k}} 2^{sJ_{N_{k-1}}} \leq \Delta_k(V_{k-1}) \leq c_2' 2^{-s\theta J_{N_{k-1}}} 2^{sJ_{N_k}}$$

by (15) and the fact that $2^{-s\theta J_{N_{k-1}}} \leq 1$ we get

$$c_1' 2^{sJ_{N_k}(1-\frac{1}{k})} \leq \Delta_k(V_{k-1}) \leq c_2' 2^{sJ_{N_k}}.$$

Hence

$$(c_2')^{-p}\left(\prod_{k=1}^{p} 2^{sJ_{N_k}}\right)^{-1} \leq m_\theta(V) \leq (c_1')^{-p}\left(\prod_{k=1}^{p} 2^{sJ_{N_k}(1-\frac{1}{k})}\right)^{-1}.$$

Recalling that by (15), $J_{N_k} > e^{J_{N_{k-1}}}$ for every k, thus

$$\lim_{p\to+\infty} \frac{p}{J_{N_p}}\left(p\frac{\log c_2'}{s\log 2} + \sum_{k=1}^{p-1} J_{N_k}\right) = 0$$

(since $\displaystyle\sum_{k=1}^{p-1} J_{N_k} \leq (p-1)J_{N_{p-1}} \leq (J_{N_{p-1}})^2$, we get $\dfrac{p\sum_{k=1}^{p-1} J_{N_k}}{J_{N_p}} \leq p(J_{N_{p-1}})^2 e^{-J_{N_{p-1}}} \leq (J_{N_{p-1}})^3 e^{-J_{N_{p-1}}} \underset{p\to+\infty}{\to} 0$). Thus there exists p_1 such that for all $p \geq p_1$,

$$(c_2')^{-p}\left(\prod_{k=1}^{p-1} 2^{sJ_{N_k}(1-\frac{1}{k})}\right)^{-1} \geq 2^{-\frac{s}{p}J_{N_p}}$$

hence, for all $p \geq p_1$,

$$2^{-sJ_{N_p}(1+\frac{1}{p})} \leq m_\theta(V).$$

As previously, $\displaystyle\lim_{p\to+\infty} \frac{p}{J_{N_p}}\left(p\frac{\log c_1'}{s\log 2} + \sum_{k=1}^{p-1} J_{N_k}(1-\frac{1}{k})\right) = 0$. Thus there exists p_2 such that for all $p \geq p_2$

$$(c_1')^{-p}\left(\prod_{k=1}^{p-1} 2^{sJ_{N_k}(1-\frac{1}{k})}\right)^{-1} \leq 2^{\frac{s}{p}J_{N_p}}$$

hence for all $p \geq p_2$

$$m_\theta(V) \leq 2^{-sJ_{N_p}(1-\frac{2}{p})}.$$

To prove (20), remark that for all $V \in F_p$, $|V| \approx 2^{-\theta J_{N_p}}$ (where $\approx$ means that the ratio of the two quantities is bounded from below and above by two positives constants independents of p). Then, $p = o(|\log |V||)$. Thus (19) yields (20). $\square$

Now we extend (20) to all Borel subsets of K.

Lemma 3.6. *There is two positive sequences $(\eta_p)_p$, decreasing to* 0 *and a constant $C > 0$ such that for any Borel set $B \subset K$ with $|B| \leq \eta_p$ we have*

$$m_\theta(B) \leq |B|^{\frac{s}{\theta}-\frac{2}{p-1}}. \tag{21}$$

Proof. We follow the same ideas of [2]. Let $\eta_p = 2^{-J_{N_p}}$. Let B be a Borel set such that $B \subset K$ with $|B| < \eta_p$. Let $q \geq p+1$ the unique integer such that

$$2^{-J_{N_q}} \leq |B| < 2^{-J_{N_{q-1}}}.$$

Since for all $V, V' \in F_{q-1}$, $V = \overline{B}(x_{\mathbf{j}}, 2^{-\theta J_{N_{q-1}}}) \cap K$, $V' = \overline{B}(x_{\mathbf{j}'}, 2^{-\theta J_{N_{q-1}}}) \cap K$, we have $\overline{B}(x_{\mathbf{j}}, 2^{-J_{N_{q-1}}}) \cap \overline{B}(x_{\mathbf{j}'}, 2^{-J_{N_{q-1}}}) = \emptyset$ then B intersect at most C elements of F_{q-1}, where C is a constant independent of p.

Let us distinguish two cases

• $2^{-\theta J_{N_{q-1}}} \leq |B| < 2^{-J_{N_{q-1}}}$: if B dont intersect no one of F_{q-1} then $m_\theta(B) = 0$. Otherwise, denoting by V any one of F_{q-1} intersecting B. Using (19) we have

$$\begin{aligned} m_\theta(B) &\leq C m_\theta(V) \leq C 2^{-sJ_{N_{q-1}}(1-\frac{2}{q-1})} \\ &\leq C\,|B|^{\frac{s}{\theta}(1-\frac{2}{q-1})} \leq |B|^{\frac{s}{\theta}(1-\frac{2}{p-1})}. \end{aligned}$$

• $2^{-J_{N_q}} \leq |B| < 2^{-\theta J_{N_{q-1}}}$: Let $V \in F_{q-1}$ that intersect B (if there is no such one then $m_\theta(B) = 0$). We have proved that for any $U \in F_q$, such that $U \subset V$,

$$m_\theta(U) = \frac{m_{\theta(V)}}{\Delta_q(V)}.$$

By Lemma 3.4 we have $\Delta_q(V) \geq c_1' 2^{-s\theta J_{N_{q-1}}} 2^{sJ_{N_q}}$, then

$$m_\theta(U) \leq \frac{1}{c_1'} m_\theta(V) 2^{s\theta J_{N_{q-1}} - sJ_{N_q}}. \tag{22}$$

In the other hand, since B is within a ball of side length $C\,|B|$, where $C \geq \max\{2, |K|\}$, the number of elements of F_q that intersecting B is less than $c\,|B|^s\,2^{sJ_{N_q}}$.

Indeed, $B \subset B'$ where B' is a ball of side length $C\,|B|$. If $U = \overline{B}(x_{\mathbf{j}}, 2^{-\theta J_{N_q}}) \cap K$ such that $U \cap B \neq \emptyset$, then $K_{\mathbf{j}} \subset B'$ (since $|K_{\mathbf{j}}| \leq |K|\,2^{-J_{N_q}}$ and then $K_{\mathbf{j}} \subset B(x_{\mathbf{j}}, |K|\,2^{-J_{N_q}}) \subset B'$). Denote by L the set of $U \in F_p$ such that $U \cap B = \emptyset$ and S the set of $\mathbf{j} \in I(2^{-J_{N_q}})$ such that $K_{\mathbf{j}} \subset B'$, for all such $\mathbf{j}$ we have $K_{\mathbf{j}} \subset B' \cap K$. We have $\sharp L \leq \sharp S$. But

$$\lambda\left(\bigcup_{\mathbf{j}\in S} K_\alpha\right) \leq \lambda\left(B' \cap K\right) \leq c\,|B'|^s \leq c'\,|B|^s$$

since

$$\lambda\left(\bigcup_{\mathbf{j}\in S} K_\alpha\right) = \sum_{\mathbf{j}\in S} \lambda\left(K_{\mathbf{i}}\right) \geq c'' \sharp S 2^{-sJ_{N_q}}$$

we get

$$\sharp L \le \sharp S \le c\,|B|^s\, 2^{sJ_{N_q}}.$$

Hence, gathering all the estimation above and the fact that $|B|^{-\frac{1}{\theta}} > 2^{J_{N_{q-1}}}$ we get

$$\begin{aligned}
m_\theta(B) &\le \sum_{U\in L} m_\theta(U) \\
&\le c'\,|B|^s\, 2^{sJ_{N_q}} m_\theta(V) 2^{s\theta J_{N_{q-1}}+sJ_{N_q}} \le C\,|B|^s\, 2^{s\theta J_{N_{q-1}}} m_\theta(V) \\
&\le C\,|B|^s\, 2^{s\theta J_{N_{q-1}}} 2^{-sJ_{N_{q-1}}(1-\frac{2}{p-1})} \le C\,|B|^s\, 2^{sJ_{N_{q-1}}(\theta-1+\frac{2}{q-1})} \\
&\le C\,|B|^s\, |B|^{-\frac{1}{\theta}s(\theta-1+\frac{2}{q-1})} \le C\,|B|^{\frac{s}{\theta}(1-\frac{2}{p-1})}.
\end{aligned}$$

□

Already we have all the ingredients to finish the proof of Theorem 1.1 with the same way as in [2] by considering the sets

$$\widetilde{E}_\mu(h) = \{x \in K :\ h_\mu(x) \le h\} = \bigcup_{h'\le h} E_\mu(h).$$

For safe completeness we recover their idea.

Proposition 3.4. *For any $h \in]0, s]$, $d_\mu(h) = h$.*

Proof. Let $h \in]0, s]$, and $\theta = \dfrac{s}{\theta}$.

A standard result claim that for all $h \ge 0$,

$$\dim_{\mathcal{H}} \widetilde{E}_\mu(h) \le \min\{h, d\}. \tag{23}$$

Then, by Proposition 3.2, $\Lambda_\theta \subset \widetilde{E}_\mu(h)$. Let us write

$$\Lambda_\theta = (\Lambda_\theta \cap E_\mu(h)) \bigcup \left(\bigcup_{n\ge 1} \Lambda_\theta \cap \widetilde{E}_\mu(h - \frac{1}{n}) \right).$$

Now, consider the measure m_θ provided by Theorem 3.1 which is supported by the Cantor set $\mathcal{C}_\theta \subset \Lambda_\theta$. We have then

$$m_\theta(\Lambda_\theta) \ge m_\theta(\Lambda_\theta) > 0.$$

By (23), for any $n \ge 1$, $\dim_{\mathcal{H}} \left(\Lambda_\theta \cap \widetilde{E}_\mu(h - \frac{1}{n}) \right) \le h - \dfrac{1}{n} < h$. Since $\dim_{\mathcal{H}} m_\theta \ge \dfrac{s}{\theta} = h$, then by property (13) we deduce that $m_\theta \left(\Lambda_\theta \cap \widetilde{E}_\mu(h - \frac{1}{n}) \right) = 0$.

Then, we get $m_\theta(\Lambda_\theta) = m_\theta\,(\Lambda_\theta \cap E_\mu(h)) > 0$. Hence, again by property (13)

$$\dim_{\mathcal{H}} E_\mu(h) \ge \dim_{\mathcal{H}} \Lambda_\theta \cap E_\mu(h) \ge \frac{s}{\theta} = h.$$

Finally, the upper bound results from the inclusion $E_\mu(h) \subset \widetilde{E}_\mu(h)$. This finish the proof of the Proposition 3.4. □

It remains the case $h = 0$.

Let $x_0 \in K$ any fixed point. By Theorem 2.1, there exists a G_δ set $\Omega(x_0)$ of $\mathcal{M}(K)$ such that for all $\mu \in \Omega(x_0)$, $h_\mu(x_0) = 0$. Thus, for all $\mu \in \Omega(x_0)$, $x_0 \in E_\mu(0)$. Hence, for all $\mu \in \Omega(x_0)$, $E_\mu(0) \neq \emptyset$. Thus, for all $\mu \in \Omega(x_0)$, $\dim_{\mathcal{H}} E_\mu(0) \geq 0$. As already we have the upper bound, then for all $\mu \in \Omega(x_0)$, $\dim_{\mathcal{H}} E_\mu(0) = 0$.

Consider $\Omega = \Omega' \cap \Omega'' \cap \Omega(x_0)$. Ω is a G_δ set of $\mathcal{M}(K)$ and for all $\mu in \Omega$, μ satisfy all the points of Theorem 1.1.

It remains for us to show that any $\mu \in \Omega$ satisfies the multifractal formalism.

Let $\mu \in \mathcal{M}(K)$. We denote by $N_j(K)$, the number of cubes of $\mathcal{G}_j$ that intersect K. By the concavity of $t \mapsto t^q$, for $q \in [0,1]$, we have for any $q \in [0,1]$

$$\begin{aligned}\sum_{Q\in\mathcal{G}_j,\mu(Q)\neq 0} \mu(Q)^q &= \sum_{Q\in\mathcal{G}_j,K\cap Q\neq\emptyset} \mu(Q)^q \\ &\leq N_j(K)\left(\frac{1}{N_j(K)}\sum_{Q\in\mathcal{G}_j,K\cap Q\neq\emptyset}\mu(Q)\right)^q = N_j(K)^{1-q}.\end{aligned}$$

Thus, for all $q \in [0,1]$

$$\begin{aligned}\tau_\mu(q) = \liminf_{j\to+\infty} -\frac{\log \sum_{Q\in\mathcal{G}_j,\mu(Q)\neq 0}\mu(Q)^q}{j\log 2} &\geq (q-1)\limsup_{j\to\infty}\frac{\log N_j(K)}{j\log 2} \\ &= (q-1)\overline{\dim_{\mathcal{B}}}(K) = (q-1)s.\end{aligned}$$

Let $\mu \in \Omega$. From (12) it follows that for all $h \in [0,s]$,

$$d_\mu(h) = h \leq (\tau_\mu)^*(h) \leq \inf_{q\in[0,1]}(qh - \tau_\mu(q)). \tag{24}$$

Hence, for all $h \in [0,s]$, for all $q \in [0,1]$, $\tau_\mu(q) \leq (q-1)h$. In particular, for $h = s$, for all $q \in [0,1]$, $\tau_\mu(q) \leq (q-1)s$. As we already have the lower bound, we conclude that for all $q \in [0,1]$, $\tau_\mu(q) = (q-1)s$. Then, for all $h \in [0,s]$,

$$\inf_{q\in[0,1]}(qh - \tau_\mu(q)) = h = d_\mu(h).$$

Thus, the inequalities of (24) turn to be equalities. Hence, for all $h \in [0,s]$,

$$d_\mu(h) = h = (\tau_\mu)^*(h).$$

□

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